On Multipliers of Spaces of Harmonic Functions in the Unit Ball of $\mathbb{R}^n$

Romi Shamoyan[1], Ali Abkar[2]

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[1]Department of Mathematics, Bryansk Technical University, Bryansk, Russia, email: rshamoyan@gmail.com
and
[2]Department of Mathematics, Imam Khomeini International University, Qazvin 34149, Iran; email: abkar@ikiu.ac.ir

Abstract. We completely describe spaces of multipliers of certain harmonic function spaces of Bergman type in $\mathbb{R}^n$. This is the first sharp result of this kind for Bergman type mixed norm spaces of harmonic functions in unit ball of R^n

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1 Introduction

Let B^n be the unit ball in Euclidean space $\mathbb{R}^n$: that is

$$B^n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} < 1 \right\},$$

and $S^n = \partial B^n$ be the unit sphere in $\mathbb{R}^n$; $S^n = \{x \in \mathbb{R}^n : |x| = 1\}$. Let $dm_n(x)$ and dx' be the normalized Lebesgue measures in B^n and S^n respectively. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, its projection in S^n is defined as usual by $x = x'|x|$.
In [1] the following Bergman spaces $A^p_\alpha(B^n)$ of harmonic functions in B^n were defined:

$$A^p_\alpha(B^n) = \{f \in h(B^n) : \|f\|_{p,\alpha} = \left(\int_0^1 \int_{S^n} |f(rx')|^p (1-r)^\alpha r^{n-1} dr dx'\right)^{1/p} < \infty\}$$

and for $p = \infty$ and $0 \leq \alpha < \infty$ we define

$$A^\infty_\alpha(B^n) = \{f \in h(B^n) : \|f\|_{\infty,\alpha} = \sup_{x \in B^n} |f(x)|(1-|x|)^\alpha < \infty\}$$

where $h(B^n)$ is the space of all harmonic functions in B^n. Note that A^p_α is a Banach space for $p \geq 1$, and a complete metric space for $0 < p < 1$ (see [1]). In the reference [1], the integral representation was provided and the existence of bounded projections were proved from Lebesgue $L^p_\alpha(B^n)$ classes to $A^p_\alpha(B^n)$ spaces.
We will define multipliers of these classes and describe spaces of multipliers of such type spaces. For this objective, we will need several additional objects and definitions to formulate the main result of this paper.
It is well-known that every harmonic function $f(x)$ in the unit ball B^n can be represented as $f(x) = \sum_{k \geq 0} r^k c_k y^{(k)}(x')$ or

$$f(x) = \sum_{k \geq 0} r^k \left(\sum_{g=1}^{d_k} c_k^{(j)} y_j^{(k)}(x')\right), \qquad x = rx' \in B^n,$$

where $y_j^{(k)}(x)$ is a spherical harmonic function (see [1], [2], [3]). It is well-known that this system of functions form an orthonormal system on the unit sphere S^n of $L^2(S^n)$, by taking unions of $y_j^{(k)}(x)$ by k, (see [1]). We will need the following vital kernel function of $2n$ variables $Q_m(x,y)$, $x = rx', y = \rho y'$, $m > 0, \rho > 0, r > 0$ (see [1]):

$$Q_m(x,y) = 2 \sum_{k \geq 0} \frac{\Gamma(m+1+k+n/2}{\Gamma(m+1)\Gamma(m+n/2)} |x|^k |y|^k Z_{x'}^{(k)}(y')$$

where $Z^{(k)}_{x'}(y')$ is a Zonal harmonic function(see[1]).

Definition. We say that a sequence of complex numbers $c_k = \{c_k^{(j)} : j = 1, ..., d_k\}$, $k \geq 0$ is a multiplier from X to Y, where X and Y are subspaces of $h(B^n)$ if for any $f \in X$ with

$$f(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} b_k^{(j)} y_j^{(k)}(x') = \sum_{k\geq 0} r^k \left(b_k y^k(x)\right)$$

we have the function $g(x) = \sum_{k\geq 0} r^k b_k c_k y^k(x\prime) \in Y$ where $x = rx' \in B^n$. Indeed

$$g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} b_k^{(j)} c_k^{(j)} y_j^{(k)}(x').$$

In this case we will write $\{c_k\} \in M_H(X, Y)$.

We also define general mixed norm classes of harmonic functions in B^n as follows: for $0 < p, q < \infty$, $-1 < \alpha < \infty$, we set

$$A^{p,q}_\alpha(B^n) = \{f \in h(B^n) : \|f\|^p_{p,q,\alpha} < \infty\}$$

where

$$\|f\|^p_{p,q,\alpha} = \int_0^1 \left(\int_{S^n} |f(|x|x')|^q dx'\right)^{p/q} (1 - |x|^2)^\alpha |x^{n-1}|d|x|.$$

Note these spaces are Banach spaces for $\min(p, q) > 1$ and are complete metric spaces for $\max(p, q) \leq 1$. Note also that for $0 < p < \infty$, we have $A^{p,p}_\alpha(B^n) = A^p_\alpha(B^n)$.

For two harmonic functions f, g with

$$f(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^{(j)} y_j^{(k)}(x');$$

and

$$g(x) = \sum_{k\geq 0} \rho^k \sum_{j=1}^{d_k} b_k^{(j)} y_j^{(k)}(x')$$

we define the convolution of f and g with

$$(f*g)(x)=\sum_{k\geq 0}r^k\sum_{j=1}^{d_k}c_k^{(j)}b_k^{(j)}y_j^{(k)}(x');\quad x=rx'.$$

The function

$$P_x(y')=P(x,y')=\sum_{k\geq 0}r^kZ_{x'}^{(k)}(y')=\sum_{k\geq 0}r^k\left(\sum_{j=1}^{d_k}y_j^{(k)}(y')y_j^{(k)}(x\prime)\right)$$

is a Poisson kernel(see[1]), and finally $(\Lambda_{m+1}f)(x)$ is a known fractional derivative of $f(x)$ (see [1]):

$$(\Lambda_{m+1}f)(rx')=\sum_{k\geq 0}r^kc_ky^{(k)}(x')\frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)}.$$

In the next section we shall use all these terminologies in more details.

2 Main Result

In this main section we will provide a complete description for the multipliers of $A_\alpha^{p,1}$ classes with some restrictions on indexes. This is the first result of this type in harmonic spaces with mixed norm.

Theorem. Let $g(x)$ be a harmonic function in B^n and

$$g(x)=\sum_{k\geq 0}r^k\sum_{j=1}^{d_k}c_k^{(j)}y_j^{(k)}(x'),\quad x=\rho x',\rho\in(0,1),x'\in S^n.$$

Let $0<p\leq 1$, $\alpha\in(0,1)$, $m>\max\{\alpha-1,1/p-1\}$, $\beta>0$. Then the following assertions are equivalent:

1)

$$\{c_k^{(j)}:j=1,...,d_k,k\in\mathbb{Z}^+\}\in M_H(A_\alpha^{p,1},A_\beta^{p,1})$$

2)

$$\sup_{0<\rho<1}\sup_{y'\in S^n}\left(\int_{S^n}|\Lambda_{m+1}(g*P_{x'})(\rho y')|dx'\right)(1-\rho)^{m+1-\alpha+\beta}<\infty.$$

Proof. To prove Theorem, we shall need several rather elementar and partially known lemmas.

Lemma 1. The function $Q_\beta(x,y)$ can be estimated in the following way:

$$|Q_\beta(x,y)| \le \frac{C_1(1-r)^{-\beta}}{|r\rho x' - y'|^{n+[\beta]}} + \frac{C_2}{(1-r\rho)^{1+\beta}},$$

where $x = rx'$, $y = \rho y'$, $\beta > -1$, $r, \rho \in (0,1)$.

The proof of this lemma can be found in [1].

Lemma 2. Let $\alpha > -1$ and $\lambda > \alpha + 1$. Then

$$\int_0^1 (1-r)^\alpha (1-r\rho)^{-\lambda} dr \le C_{\alpha,\lambda}(1-\rho)^{\alpha-\lambda+1}; \qquad \rho \in (0,1).$$

The proof can be found in [1].

Lemma 3. Let $f, g \in h(B^n)$. Then using expansions for f and g mentioned above we have

$$\int_{S^n} (g * P_y)(rx')f(rx')dx' = \int_{S^n} \left(\sum_{k\ge 0} r^k \sum_{j=1}^{d_k} C_k^{(j)} y_j^{(k)}(y') y_j^{(k)}(x') \right)$$

$$\left(\sum_{m\ge 0} r^m \sum_{i=1}^{d_m} y_i^{(m)}(x') b_m^{(i)} dx' \right) =$$

$$= \sum_{k\ge 0} r^{2k} \sum_{j=1}^{d_k} C_k^{(j)} b_k^{(j)} y_j^{(k)}(y').$$

The proof is based on the known orthonormality properties of $y_j^{(k)}$we listed above(see[1]),we omit details.

Lemma 4. Let $m, k, n \in \mathbb{N}$. Then

$$\int_0^1 (1-R^2)^m R^{2k+n-1} dR = \frac{1}{2} \frac{\Gamma(m+1)(\Gamma(k+n/2)}{\Gamma(m+1+n/2+k)}.$$

The proof can be found in [1].

Lemma 5. Let $0 < p \le \infty$,

$$M_p(f,r) = \left(\int_{S^n} |f(rx')|^p dx\prime \right)^{1/p}, \qquad 0 < r \le 1,$$

where dx' is as we defined above the Lebesgue measure on ∂B^n. Let $0 < q \leq 1$, $-1 < \beta < \infty$. Then

$$\left(\int_0^1 M_p(f,|y|)\frac{(1-|y|)^{\beta}}{(1-|x||y|)^{\beta+1}}|y|^{n-1}d|y|\right)^q \leq C_q\left(\int_0^1 \frac{M_p^q(f,|y|)(1-|y|)^{\beta q+q-1}}{(1-|x||y|)^{(\beta+1)q}}|y|^{n-1}d|y|\right).$$

Proof. For each $f \in h(B^n)$, we have $M_p(f,r_1) \leq M_p(f,r_2)$ whenever $r_1 \leq r_2$. Since $0 < q \leq 1$, it follows that

$$\begin{aligned} J &= \left(\int_0^1 M_p(f,|y|)\frac{(1-|y|)^{\beta}}{(1-|x||y|)^{\beta+1}}|y|^{n-1}dy\right)^q \\ &\leq C_1\sum_{k\geq 0} M_p^q(f,1-2^{-k-1})(2^{-k\beta q})\left(\int_{1-2^{-k}}^{1-2^{-k-1}} \frac{|y|^{n-1}d|y|}{(1-|x||y|)^{\beta+1}}\right)^q \\ &\leq C_2\sum_{k\geq 0} M_p^q(f,1-2^{-k-1})(2^{-k\beta q})\left(1-|x|(1-2^{-k-1})\right)^{(\beta+1)q}2^{-kq}. \end{aligned}$$

Obviously,

$$\frac{1-|x|(1-2^{-k-1})}{1-|x|(1-2^{-k+1})} \geq \frac{1}{4}; \quad 0 < |x| \leq 1,\ k \geq 0.$$

Hence

$$\begin{aligned} J &\leq C_3\sum_{k\geq 0}\left(\int_{1-2^{-k-1}}^{1-2^{-k-2}} d|y|\right)\left(2^{-k(-1+q+\beta q)}\right) \\ &\qquad \left(M_p^q(f,1-2^{-k-1})\right)\left(1-|x|(1-2^{-k+1})\right)^{-(\beta+1)q} \\ &\leq C_4\sum_{k\geq 0}\int_{1-2^{-k-1}}^{1-2^{-k-2}} M_p^q(f,|y|)\frac{(1-|y|)^{\beta q+q+1}}{(1-|x||y|)^{(\beta+1)q}}d|y| \\ &\leq C_5\int_0^1 \frac{M_p^q(f,|y|)(1-|y|)^{q(\beta+1)-1}}{(1-|x||y|)^{(\beta+1)q}}d|y| \\ &\leq C_6\int_0^1 M_p^q(f,t)\frac{(1-t)^{q(\beta+1)-1}}{(1-|x|t)^{(\beta+1)q}}t^{n-1}dt. \end{aligned}$$

This is what we wanted to prove.

Lemma 6. Let $y' \in S^n$ be a fixed point in the unit sphere of $\mathbb{R}^n$. Let

$P(x,y') = \omega_{n-1}\frac{1-|x|^2}{|x-y'|^n}$ be the Poisson kernel (see[1]) for the unit ball B^n, $x \in B^n$, $y' \in S^n$ and ω_{n-1}^{-1} is the area of the unit sphere. If $x = rx'$, then

$$P(x,y') = \sum_{k\geq 0} r^k Z_{x'}^{(k)}(y')$$

$$= \sum_{k\geq 0} r^k \left(\sum_{j=1}^{d_k} y_j^{(k)}(y') y_j^{(k)}(x') \right) = P_{y'}(rx)$$

moreover, for $m \in \mathbb{N}$ and $x = rx'$ we have

$$\int_{S^n} (g * P_{y'})(rx')(f(rx'))dx' =$$

$$2\int_0^1 \int_{S^n} \Lambda^{m+1}(g * P_{y'})(rR\xi)(f(rR\xi))(1-R^2)^m R^{n-1} dR d\xi$$

where as above

$$((\Lambda^{m+1})f)(rx') = \sum_{k\geq 0} r^k C_k(y^k(x')) \left(\frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)} \right).$$

Proof of Lemma 6. We have to use the orthonormality of $\{y_j^{(k)}\}_{j=1}^{d_k}$ (see [1]) and Lemma 4. We have the following chain of equalities

$$2\int_0^1 \int_{S^n} \Lambda^{m+1}(g * P_{y'})(rR\xi)(f(rR\xi))(1-R^2)^m R^{n-1} dR d\xi$$

$$= 2\int_0^1 \int_{S^n} \left(\sum_{k\geq 0} (rR)^k \frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)} \sum_{j=1}^{d_k} C_k^{(j)} y_j^{(k)}(y') y_j^{(k)}(\xi) \right) \times$$

$$\left(\sum_{m\geq 0} (rR)^m \sum_{i=1}^{d_m} b_m^{(i)} y_i^{(m)}(\xi) \right) (1-R^2)^m R^{n-1} dR d\xi$$

$$= 2\int_0^1 \sum_{k\geq 0} \frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)} R^{2k} r^{2k} \sum_{j=1}^{d_k} C_k^{(j)} b_k^{(j)} y_j^{(k)}(y')(1-R^2)^m R^{n-1} dR$$

$$= \sum_{k\geq 0} r^{2k} \sum_{j=1}^{d_k} C_k^{(j)} b_k^{(j)} y_j^{(k)}(y').$$

On the other hand we have

$$\int_{S^n}(g*P_{y'})(rx')(f(rx'))dx' = \int_{S^n}\left(\sum_{k\geq 0} r^k \sum_{j=1}^{d_k} C_k^{(j)} y_j^{(k)}(y')y_j^{(k)}(x')\right)$$
$$\left(\sum_{m\geq 0} r^m \sum_{i=1}^{d_m} y_i^{(m)}(x')b_m^{(i)}\right) dx' = \sum_{k\geq 0} r^{2k}\sum_{j=1}^{d_k} C_k^{(j)} b_k^{(j)} y_j^{(k)}(y').$$

Part two is proved.For part one of lemma see[1]. Now based on these lemmas we are in a position to prove our theorem.

Proof of Theorem. Fix $y' \in S^n$. For each $y', m \in \mathbb{N}, m > \alpha - 1$ we consider a function

$$Q_m(y,x) = \sum_{k\geq 0}(r^k)\sum_{j=1}^{d_k} \rho^k \left(\frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)} y_j^{(k)}(y')\right) y_j^{(k)}(x')$$

where $x = rx', y = \rho y'$. Since $\{C_k^{(j)} : j = 1, 2, ..., d_k, k \geq 0\} \in M_H(A_\alpha^{p,1}, A_\beta^{p,1})$, we can easily verify that (see [1])

$$h_y(x) = \sum_{k\geq 0}(r^k)\left(\sum_{j=1}^{d_k} \rho^k \frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)} y_j^{(k)}(y')C_k^{(j)}\right) y_j^{(k)}(x')$$
$$= \sum_{k\geq 0}(r^k\rho^k)\left(\sum_{j=1}^{d_k} \frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)\Gamma(m+1)} C_k^{(j)} y_j^{(k)}(x')\right) y_j^{(k)}(y')$$

is in $A_\beta^{p,1}(B^n)$. Using the closed graph theorem, for each $m \in \mathbb{N}, m > p(\alpha - 1)$ we have

$$\|h_y(x)\|_{A_\beta^{p,1}} \leq \|\Lambda_{m+1}(g*P_{x'})(r\rho y')\|_{A_\beta^{p,1}} \leq C\|Q_m(y,x)\|_{A_\beta^{p,1}}.$$

Now we use the lemmas proved(Lemma1) to conclude that

$$\int_{S^n} |Q_\beta(x,y)|dx' \leq C\left(\int_{S^n} \frac{(1-r\rho)^{-\beta}}{|r\rho x' - y'|^{n+[\beta]}} + \frac{1}{(1-r\rho)^{1+\beta}}\right)$$
$$\leq C\frac{1}{(1-r\rho)^{1+\beta}}.$$

Hence for $m > (\alpha-1)p$ we have

$$\|Q_m(y,x)\|_{A^{p,1}_\alpha} = \left(\int_0^1 (\int_{S^n} |Q_m(x,y)|dx'\right)^p (1-|x|)^{\alpha p-1}|x|^{n-1}dx)^{1/p}$$
$$\leq C(\int_0^1 \left(\frac{1}{(1-|x||y|)^{(n+1)p}}(1-|x|)^{\alpha p-1}|x|^{n-1}d|x|\right)^{1/p} \leq C(1-|y|)^{-(m+1-\alpha)}.$$

We now use the fact that for $0 < r_1 \leq r_2 < 1$, $M_p(f,r_1) \leq M_p(f,r_2)$ to obtain

$$\int_{S^n} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|dx' = \left(\int_{\rho_1}^1 ... \int_{\rho_n}^1 (1-r)^{\beta p-1} r^{n-1} dr\right)^{-1/p}$$
$$\int_{\rho_1}^1 ... \int_{\rho_n}^1 (1-r)^{\beta p-1} \left(\int_{S^n} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|dx' \cdot r^{n-1}dr\right)^{1/p}$$
$$\leq C(1-\rho)^{-\beta}\|\Lambda_{m+1}(g * P_{x'})(r\rho y')\|_{A^{p,1}_\beta}$$

where $x = rx'$, $y = \rho y'$. Finally, for $m > \alpha - 1$ we have

$$\int_{S^n} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|dx' \leq C(1-\rho)^{-(m+1-\alpha+\beta)}$$

or

$$(\sup_{0<\rho<1})(\sup_{y'\in S^n})\left(\int_{S^n} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|dx'\right) \times (1-\rho)^{m+1-\alpha+\beta} < \infty.$$

We now manage to prove the reverse implication of the theorem. By defintion of convolution above we have

$$f(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^{(j)} y_j^{(k)}(x'), \quad g(y) = \sum_{m\geq 0} \rho^m \sum_{i=1}^{d_m} b_m^{(i)} y_i^{(m)}(y'),$$
$$(f * g)(x) = \sum_{k\geq 0} (r\rho)^k \sum_{j=1}^{d_k} c_k^{(j)} b_k^{(j)} y_j^{(k)}(x'); \quad x = rx', y = \rho y'$$

then it follows from lemmas we proved

$$h(\rho r x') = \int_{S^n} (g * P_{x'})(ry') f(\rho y') dy' =$$
$$= 2\int_0^1 \int_{S^n} (\Lambda^{m+1} g * P_{x'})(r\rho\xi)(1-R^2)^m R^{n-1} dR d\xi, \quad m > \alpha - 1.$$

Then we have

$$\int_{S^n} |h(\rho r x')| dx' \leq \int_0^1 \int_{S^n} \int_{S^n} |\Lambda^{m+1}(g * P_{x'})(rR\xi)|$$

$$|f(\rho R\xi)|(1-R^2)^m R^{n-1} dR d\xi dx' \leq$$

$$C \int_0^1 \sup_{\xi \in S^n} \int_{S^n} |\Lambda^{m+1}(g * P_{x'})(rR\xi)| dx'$$

$$\times \left(\int_{S^n} |f(R\xi)| d\xi \right) (1-R^2)^m R^{n-1} dR.$$

It is well-known that for each fixed $\xi \in S^n$ by subharmonicity, the function

$$u(rR\xi, m) = \int_{S^n} |\Lambda^{m+1}(g * P_{x'})(rR\xi)| dx'$$

is growing, hence $\sup_{\xi \in S^n}(u_\xi)$ is also growing , and hence we can use the lemmas above (Lemma 5) to get the following estimate:

$$\int_0^1 \left(\int_{S^n} |h(rx')| dx' \right)^p (1-r)^{\beta p - 1} r^{n-1} dr \leq$$

$$C_1 \int_0^1 \int_0^1 \sup_{\xi \in S^n} \left(\int_{S^n} |\Lambda^{m+1}(g * P_{x'})(rR\xi)| dx' \right)^p$$

$$\times \left(\int_{S^n} |(f(R\xi)| d\xi \right)^p (1-R)^{pm+p-1} R^{n-1} dR (1-r)^{\beta p - 1} r^{n-1} dr.$$

We now use the estimates $(1-R)^{pm} \leq (1-rR)^{pm}$ or $(1-R)^{pm+p-1} \leq (1-rR)^{pm+p-1}$ for $m > \frac{1-p}{p}$, and the conditions imposed in the statement of the theorem to conclude that

$$\|h\|^p_{A^{p,1}_\beta} \leq C_2 \int_0^1 \int_0^1 \left(\int_{S^n} |f(R\xi)| d\xi \right)^p (1-Rr)^{p\alpha - p\beta - p} (1-R)^{p-1}$$

$$R^{(n-1)p} dR (1-r)^{\beta p - 1} r^{n-1} dr \leq C_3 \int_0^1 \left(\int_{S^n} |f(R\xi)| d\xi \right)^p$$

$$(1-R)^{p-1} \left(\int_0^1 (1-Rr)^{p\alpha - p\beta - p} (1-r)^{\beta p - 1} dr \right) R^{(n-1)p} dR$$

$$\leq C \|f\|^p_{A^{p,1}_\alpha}, \qquad 0 < \alpha < 1,\ m \in \mathbb{N};$$

here we used Lemma 2 at the last step. Now the proof of theorem is complete. We remark finnaly for p=1 this theorem was announced many years ago in [4]

References

[1] M.M. Djrbashian, F.A. Shamoyan, *Topics in the Theory of A^p_α Spaces*, Teubner Texte zur Matematik, Leipzig 1988, Band 105.

[2] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

[3] E. M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971.

[4] R.F.Shamoyan,*Bounded projections and descriptions of multipliers in harmonic function spaces in unit ball* ,Proc.MAth. Center of Lobachevski,Kazan,1998,p.260-261,in Russian